

International Journal of Modern Physics A  
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## Fragmentation Functions of neutral mesons $\pi^0$ and $k^0$ with Laplace transform approach

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Received Day Month Year

Revised Day Month Year

With an analytical solutions of DGLAP evolution equations based on the Laplace transform method, we find the fragmentation functions (FFs) of neutral mesons,  $\pi^0$  and  $k^0$  at NLO approximation. We also calculated the total fragmentation functions of these mesons and compared them with experimental data and those from global fits. The results show a good agreements between our solutions and other models and also are compatible with experimental data.

*Keywords:* Laplace transform; Fragmentation Functions ; Natural Meson.

PACS numbers:12.38.Bx, 13.60.Hb, 13.85.Hd, 13.66.Bc

### 1. Introduction

Fragmentation process is the QCD process in which partons hadronize to colorless hadrons. In this transition, the parton fragmentation function,  $D_i^h(z, Q^2)$ , represents the probability for a parton  $i$  to fragments into a particular hadron  $h$  carrying a certain fraction of the parton energy or momentum. Therefore, these fragmentation functions (FFs) are essential inputs to study the hadron production in any processes like  $p\bar{p}$ ,  $ep$ ,  $\gamma p$  and  $\gamma\gamma$  scattering.

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Fragmentation Functions evolved with DGLAP evolution equations from a starting distribution at a defined energy scale,<sup>12</sup>

Recently we have used Laplace transformation and provided an analytical solution to DGLAP equations to calculate the proton, pion and kaon fragmentation functions.<sup>3</sup> This method had provided analytical solutions to Polarized Parton Distribution Functions (PPDFs) too.<sup>4,5</sup>

In the present paper we will use the results of this new method introduced by Block et al.<sup>6-11</sup> to calculate the neutral mesons,  $\pi^0$  and  $k^0$  fragmentation functions. Therefore, our main task here is using our solutions for charged pion and kaon fragmentation functions,<sup>3</sup> for calculating the neutral mesons,  $\pi^0$  and  $k^0$  fragmentation functions. These solutions enable us to use the neutral mesons data in a global fit, in addition to all experimental data for total fragmentation functions of charged mesons, to determination of FFs.

The paper is organized as follows. In Section 2 we review the Laplace transform method of non-singlet, singlet and gluon DGLAP evolution equations for extracting the fragmentation functions. These solutions led us to  $\pi^+$  and  $k^+$  FFs. Then, in Section 3 we utilize the charge conjugation symmetry to calculate the fragmentation functions of  $\pi^-$  and  $k^-$ . This led us to natural mesons fragmentation functions. Finally in section 4 we calculated the fragmentation functions of neutral mesons,  $\pi^0$  and  $k^0$  and also compared them with available experimental data<sup>25-28</sup> and those from global fits.<sup>12-16</sup>

## 2. Fragmentation Functions via decoupling of the DGLAP evolution equations by Laplace transform method

### 2.1. Non-singlet case:

The fragmentation of valence quarks into hadrons defined the non-singlet fragmentation functions. The evolution of non-singlet fragmentation function are given by DGLAP evolution equations at NLO approximation as:

$$\frac{4\pi}{\alpha_s(Q^2)} \frac{\partial D_{ns}}{\partial \ln(Q^2)}(z, Q^2) = D_{ns} \otimes \left[ P_{qq}^{LO,ns} + \frac{\alpha_s(\tau)}{4\pi} P_{qq}^{NLO,ns} \right](z, Q^2). \quad (1)$$

where

$$D_{ns}^h(z, Q^2) = D_q^h(z, Q^2) - D_{\bar{q}}^h(z, Q^2) \quad (2)$$

The  $\otimes$  symbol in Eq. (1) refers to the convolution integral. In the new method introduced by Block et al.<sup>6-11</sup>, The DGLAP evolution equations can be solved by Laplace transform approach. To summarize, by introducing two variable  $\nu \equiv \ln(\frac{1}{z})$  and  $\tau \equiv \frac{1}{4\pi} \int_{Q_0^2}^{Q^2} \alpha_s(Q'^2) d\ln Q'^2$ , and two Laplace transforms from  $\nu$  space to  $s$  space and from  $\tau$  space to  $U$  space, the DGLAP evolution equations can be solved iteratively by a set of convolution integrals which are related to initial input Fragmentation Functions at scale of  $Q_0^2$ . Two inverse Laplace transforms will take

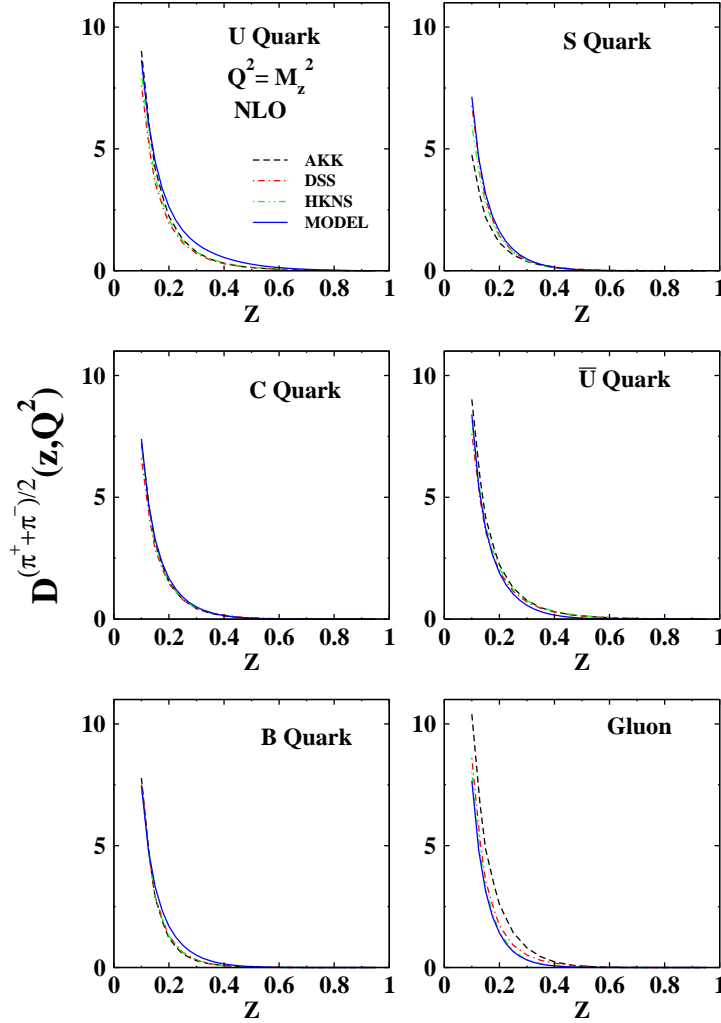
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Fig. 1. Fragmentation functions of natural pion and comparison with AKK, DSS and HKNS global fits.

us back to the usual space  $(z, Q^2)$ .<sup>3</sup> We defined  $zD_{ns}(z, Q^2) = F_{ns}(z, Q^2)$ , and find the solution of non- singlet DGLAP evolution equation, Eq. (1) in  $s$  space as<sup>3</sup> :

$$f_{ns}(s, \tau) = e^{\tau \Phi_{ns}(s)} f_{ns0}(s), \quad (3)$$

where

$$\Phi_{ns}(s) \equiv \Phi_{ns}^{LO}(s) + \frac{\tau_2}{\tau} \Phi_{ns}^{NLO}(s), \quad (4)$$

$\Phi_{ns}^{LO}(s)$  and  $\Phi_{ns}^{NLO}(s)$  are the Laplace transform of non- singlet splitting func-

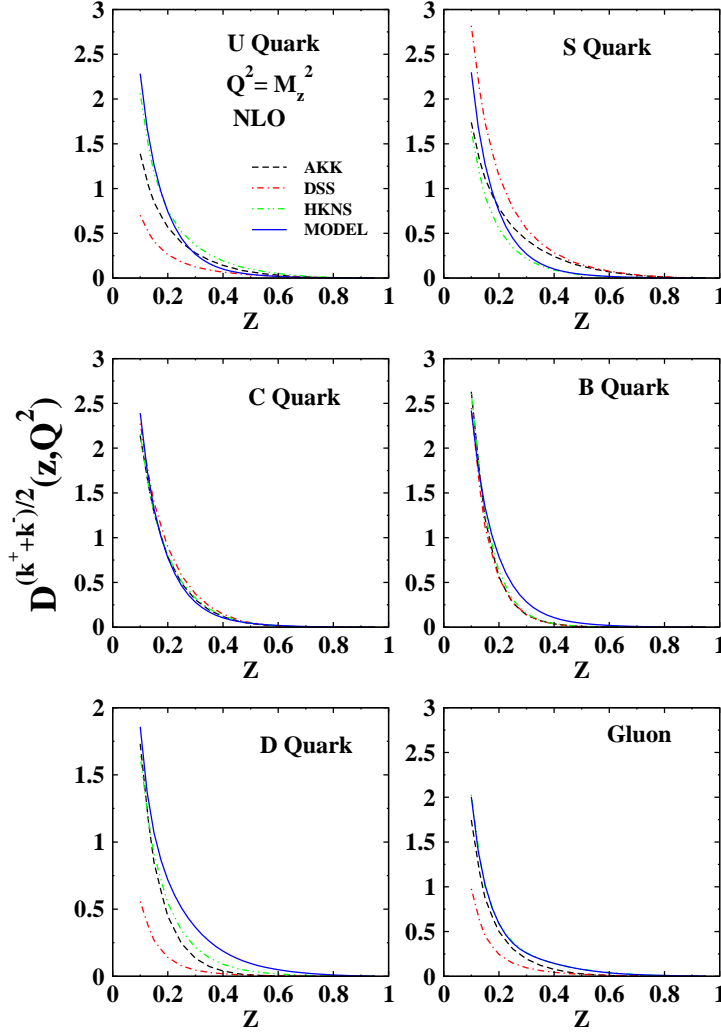


Fig. 2. Fragmentation functions of natural kaon and comparison with AKK, DSS and HKNS global fits.

tions and are given in Appendix. A of.<sup>3</sup> The  $\tau_2$  parameter is defined as

$$\tau_2 \equiv \frac{1}{4\pi} \int_0^\tau \alpha_s(\tau') d\tau' = \frac{1}{(4\pi)^2} \int_{Q_0^2}^{Q^2} \alpha_s^2(Q'^2) d \ln Q'^2, \quad (5)$$

$f_{ns0}(s)$  in Eq. (3) is the Laplace transform of initial valence quark fragmentation functions at  $Q_0 = 4.5 \text{ GeV}$ . They are selected from HKNS code<sup>12</sup> to be sure about our analytical solutions of DGLAP evolution equation. Finally, with an inverse Laplace transform of Eq. (3),<sup>11</sup> one can derive the valence quark fragmentation functions in  $(z, Q^2)$  space.

## 2.2. The singlet and gluon case:

At NLO approximation, the singlet and gluon fragmentation functions are given by these two coupled DGLAP evolution equations:

$$\frac{4\pi}{\alpha_s(Q^2)} \frac{\partial D_s}{\partial \ln Q^2}(z, Q^2) = D_s \otimes \left( P_{qq}^0 + \frac{\alpha_s(Q^2)}{4\pi} P_{qq}^1 \right) (z, Q^2) + D_g \otimes \left( P_{gq}^0 + \frac{\alpha_s(Q^2)}{4\pi} P_{gq}^1 \right) (z, Q^2), \quad (6)$$

$$\frac{4\pi}{\alpha_s(Q^2)} \frac{\partial D_g}{\partial \ln Q^2}(z, Q^2) = D_s \otimes \left( P_{qg}^0 + \frac{\alpha_s(Q^2)}{4\pi} P_{qg}^1 \right) (z, Q^2) + D_g \otimes \left( P_{gg}^0 + \frac{\alpha_s(Q^2)}{4\pi} P_{gg}^1 \right) (z, Q^2). \quad (7)$$

where the singlet fragmentation function,  $D_s^h(z, Q^2)$ , is defined as

$$D_s^h(z, Q^2) = \sum_{q=u,d,s,c,b} [D_q^h(z, Q^2) + D_{\bar{q}}^h(z, Q^2)] \quad (8)$$

By definition of  $zD_s(z, Q^2) \equiv F_s(z, Q^2)$  and  $zD_g(z, Q^2) \equiv G(z, Q^2)$ , the solutions of these coupled DGLAP evolution equations in Laplace  $(s, U)$  space can be calculated as:<sup>3</sup>

$$[U - \Phi_f(s)] \mathcal{F}(s, U) - \Theta_g(s) \mathcal{G}(s, U) = f_0(s) + a_1 [\Phi_f^{NLO}(s) \mathcal{F}(s, U + b_1) + \Theta_g^{NLO}(s) \mathcal{G}(s, U + b_1)], \quad (9)$$

$$-\Theta_f(s) \mathcal{F}(s, U) + [U - \Phi_g(s)] \mathcal{G}(s, U) = g_0(s) + a_1 [\Theta_f^{NLO}(s) \mathcal{F}(s, U + b_1) + \Phi_g^{NLO}(s) \mathcal{G}(s, U + b_1)], \quad (10)$$

here the  $\mathcal{F}(s, U)$  and  $\mathcal{G}(s, U)$  are the Laplace transformed of singlet and gluon fragmentation functions in  $(s, U)$  space. Initial input fragmentation functions are denoted by  $f_0(s)$  and  $g_0(s)$ . As we mentioned before, these initial inputs are selected from HKNS code<sup>12</sup> at initial scale of  $Q_0 = 4.5 \text{ GeV}$ . The parameters of  $a_1 = 0.025$  and  $b_1 = 10.7$  are the best fit parameters to  $a(\tau) = \frac{\alpha_s(\tau)}{4\pi} \approx a_0 + a_1 e^{-b_1 \tau}$  at NLO approximation.<sup>6</sup> The functions  $\Phi_{f,g}$  and  $\Theta_{f,g}$  specified the laplace transformation of splitting functions and can be found in<sup>3</sup> and also given in Appendix A:

$$\Phi_f(s) \equiv \Phi_f^{LO}(s) + a_0 \Phi_f^{NLO}(s), \quad \Phi_g(s) \equiv \Phi_g^{LO}(s) + a_0 \Phi_g^{NLO}(s), \quad (11)$$

$$\Theta_f(s) \equiv \Theta_f^{LO}(s) + a_0 \Theta_f^{NLO}(s), \quad \Theta_g(s) \equiv \Theta_g^{LO}(s) + a_0 \Theta_g^{NLO}(s), \quad (12)$$

With the initial input functions for  $f_0(s)$  and  $g_0(s)$ , their evolved solutions in the Laplace  $s$  space are given by<sup>10</sup>

$$\begin{aligned} f(s, \tau) &= k_{ff}(s, \tau) f_0(s) + k_{fg}(s, \tau) g_0(s) \\ g(s, \tau) &= k_{gg}(s, \tau) g_0(s) + k_{gf}(s, \tau) f_0(s) \end{aligned} \quad (13)$$

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where the  $k$ 's in Eq. (13) are given in Appendix. B of Ref.<sup>3</sup> for the first iteration. Finally, the singlet and gluon fragmentation functions in  $(z, Q^2)$  space can be calculated with known inverse laplace transform.<sup>11</sup> Our results in Ref.<sup>3</sup> show a nice agreement between these analytical solution and other global fits results for charged mesons  $\pi^+$  and  $k^+$ .

### 3. Natural pions and kaons fragmentation functions and the role of charge conjugation symmetry

The fragmentation function of total sea quarks is defined as follows

$$D_s(z, Q^2) - D_{ns}(z, Q^2) = D_{\bar{q}}(z, Q^2) \quad (14)$$

Where  $D_{\bar{q}}(z, Q^2)$  is

$$D_{\bar{q}}(z, Q^2) = 2D_{\bar{u}}(z, Q^2) + 2D_{\bar{d}}(z, Q^2) + 2D_s(z, Q^2) + 2D_c(z, Q^2) + 2D_b(z, Q^2), \quad (15)$$

Because the heavier sea quarks can produce hadrons with higher probability, we simply parameterized the fraction of different kind of sea quarks fragmentation functions as their mass ratio and then we have:<sup>3</sup>

$$D_{quark}(z, Q^2) = \frac{D_{\bar{q}}(z, Q^2)}{BA}, \quad (16)$$

The parameters of  $A$  and  $B$  are summarized in Table 1 in.<sup>3</sup> We also used these flavor symmetries between different kinds of fragmentation functions in  $\pi^+$ ,  $K^+$ :<sup>12</sup>

$$\begin{aligned} D_{\bar{u}}^{\pi^+}(z, Q^2) &= D_d^{\pi^+}(z, Q^2) \neq D_s^{\pi^+}(z, Q^2) \\ D_u^{\pi^+}(z, Q^2) &= D_{\bar{d}}^{\pi^+}(z, Q^2) \\ D_s^{\pi^+}(z, Q^2) &= D_{\bar{s}}^{\pi^+}(z, Q^2) \\ D_c^{\pi^+}(z, Q^2) &= D_{\bar{c}}^{\pi^+}(z, Q^2) \\ D_b^{\pi^+}(z, Q^2) &= D_{\bar{b}}^{\pi^+}(z, Q^2) \end{aligned} \quad (17)$$

$$\begin{aligned} D_{\bar{u}}^{K^+}(z, Q^2) &\neq D_d^{K^+}(z, Q^2) \neq D_s^{K^+}(z, Q^2) \\ D_d^{K^+}(z, Q^2) &= D_{\bar{d}}^{K^+}(z, Q^2) \\ D_c^{K^+}(z, Q^2) &= D_{\bar{c}}^{K^+}(z, Q^2) \\ D_b^{K^+}(z, Q^2) &= D_{\bar{b}}^{K^+}(z, Q^2) \end{aligned} \quad (18)$$

To calculate the natural mesons fragmentation functions, we first used the charge conjugation symmetries related the  $\pi^+(K^+)$  fragmentation functions to those of  $\pi^-(K^-)$  to derive the  $\pi^-(K^-)$  fragmentation functions:

$$\begin{aligned} D_{\bar{q}}^{\pi^+(K^+)}(z, Q^2) &= D_q^{\pi^-(K^-)}(z, Q^2) \\ D_g^{\pi^+(K^+)}(z, Q^2) &= D_g^{\pi^-(K^-)}(z, Q^2) \end{aligned} \quad (19)$$

Finally the neutral mesons fragmentation functions can be obtained by:

$$\begin{aligned} D_i^{\pi^0}(z, Q^2) &= \frac{1}{2}[D_i^{\pi^+}(z, Q^2) + D_i^{\pi^-}(z, Q^2)] \\ D_i^{K^0}(z, Q^2) &= \frac{1}{2}[D_i^{K^+}(z, Q^2) + D_i^{K^-}(z, Q^2)] \end{aligned} \quad (20)$$

Figures (1) and (2) show the results of fragmentation functions of neutral pions and kaons at  $Q^2 = M_z^2 \text{GeV}^2$ . We also compared our results with those of AKK, DSS and HKNS codes<sup>12, 15, 16</sup> to be sure about our analytical solutions.

#### 4. Total Fragmentation Functions

According to the factorization theorem,<sup>22</sup> the total fragmentation function can be expressed in terms of the partonic hard scattering cross sections and the non-perturbative fragmentation functions as:

$$F^H(z, Q^2) = \frac{1}{\sigma_{tot}} \frac{d\sigma^h}{dz}(e^-e^+ \rightarrow HX)(z, Q^2) = \sum C_i(z, Q^2) \otimes D_i^H(z, Q^2) \quad (21)$$

where,  $\sigma_{tot}$  is the total hadronic cross section.<sup>24</sup> The  $C_i(z, Q^2)$  is the Wilson coefficient function related to the partonic cross section  $e^-e^+ \rightarrow q\bar{q}$  and calculated in perturbative QCD as:<sup>14, 23</sup>

$$C_q^1(z) = C_F \left[ (1+z^2) \left( \frac{\ln(1-z)}{1-z} \right)_+ - \frac{3}{2} \frac{1}{(1-z)_+} + 2 \frac{1+z^2}{1-z} \ln(z) + \frac{3}{2}(1-z) + \left( \frac{3}{2}\pi^2 - \frac{9}{2} \right) \delta(1-z) \right], \quad (22)$$

$$C_g^1(z) = 2C_F \left[ \frac{1+(1-z)^2}{z} \ln(z^2(1-z)) - 2 \frac{1-z}{z} \right], \quad (23)$$

$$C_q^L(z) = C_F, \quad (24)$$

$$C_g^L(z) = 4C_F \frac{(1-z)}{z}. \quad (25)$$

where  $C_F = \frac{4}{3}$ . The total fragmentation functions of natural pion,  $\pi^0$  and kaon,  $K^0$  are shown in Fig. (3) and Fig. (4). We compared our result with those from HKNS global fit and also with data from TASSO, ALEPH, TOPAZ and OPAL experiments.<sup>25-28</sup> The agreement between experimental data and our model is quite reasonable. This means that our analytical solutions are correct.

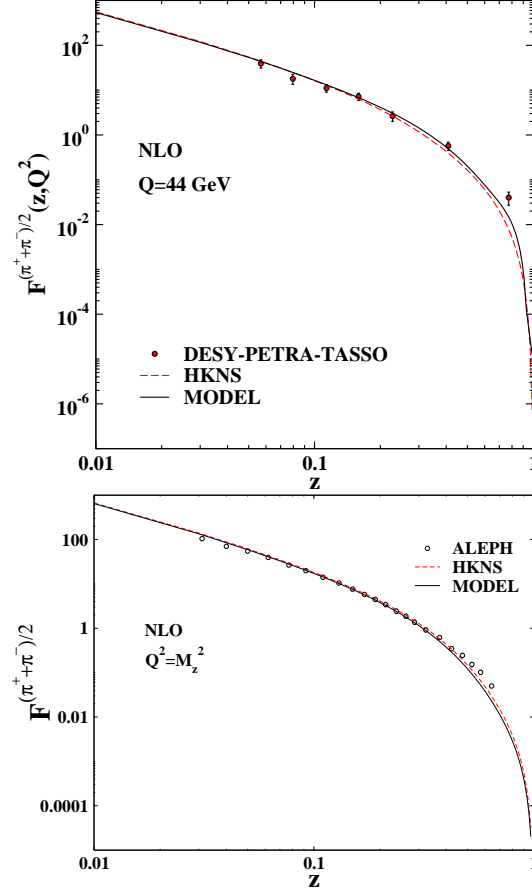


Fig. 3. Total fragmentation functions of natural pion and comparison with experimental data from TASSO and ALEPH Collaborations<sup>25, 26</sup> at  $Q = 44 \text{ GeV}$  and  $Q^2 = M_z^2$ . We also compared our results with HKNS global fit.

## 5. Conclusions and Remarks

Using the analytical solutions to DGLAP evolution equation, based on the Laplace transforms, we find the fragmentation functions of the neutral pions and kaons. Finding these solutions enable us to use the natural mesons experimental data for total fragmentation functions in a global fit to determination of fragmentation functions. This technique has the facility that the analytical solution of the fragmentation functions are obtained more strictly by using the related kernels and the calculations controlled in a better way. We have used the HKNS code for initial input fragmentation functions to be sure about our analytical solutions.

Our results for natural pions and kaons are compared with those from global fits and also with experimental data and there is a reasonable agreements between them.



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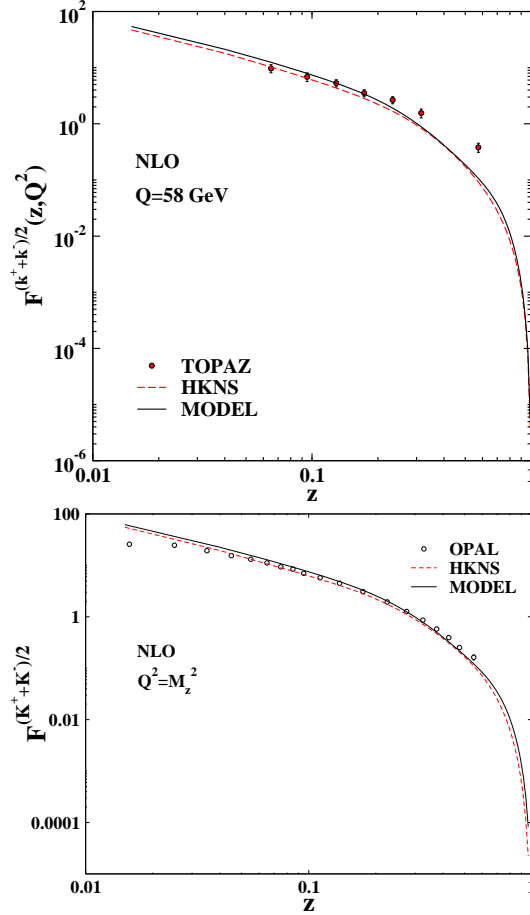


Fig. 4. Total fragmentation functions of natural kaon and proton and comparison with experimental data from TOPAZ and OPAL Collaborations<sup>27, 28</sup> at  $Q = 44 \text{ GeV}$  and  $Q^2 = M_z^2$ . We also compared our results with HKNS global fit.

### Acknowledgment

This work is supported by Ferdowsi University of Mashhad under grant 2/39420(1394/11/04).

### Appendix A

We present here the results for the Laplace transforms of splitting functions, denoted by  $\Phi^{LO, NLO}$  and  $\Theta^{LO, NLO}$  at the NLO approximation.

$$\begin{aligned}
\Phi_f^{LO}(s) &= 4 - \frac{8}{3} \left( \frac{1}{s+1} + \frac{1}{s+2} + 2(\psi(s+1) + \gamma_E) \right), \\
\Theta_g^{LO}(s) &= \frac{16}{3} n_f \left( \frac{2}{s} - \frac{2}{s+2} + \frac{2s+3}{s} \right), \\
\Theta_f^{LO}(s) &= \frac{1}{s+1} - \frac{2}{s+2} + \frac{2}{s+3}, \\
\Phi_g^{LO}(s) &= \frac{33-2n_f}{3} + 12 \left( \frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2} - \frac{1}{s+3} - \psi(s+1) - \gamma_E \right),
\end{aligned}$$

$$\begin{aligned}
\Phi_{nsqq}^{NLO} = & \\
& C_F T_f \left( -\frac{2}{3(s+1)^2} - \frac{2}{9(s+1)} - \frac{2}{3(s+2)^2} + \frac{22}{9(s+2)} + \frac{4}{3} \psi'(s+1) \right) + \\
& C_F^2 \left( \frac{5}{(s+1)^3} + \frac{5}{(s+1)^2} - \frac{5}{s+1} + \frac{5}{(s+2)^3} + \frac{3}{(s+2)^2} + \frac{5}{s+2} \right. \\
& \quad \left. - \frac{2}{(s+1)^2} \left( \gamma_E + \frac{1}{s+1} \psi(s+1) - (s+1) \psi'(s+2) \right) \right. \\
& \quad \left. - \frac{2}{(s+2)^2} \left( \gamma_E + \frac{1}{s+2} \psi(s+2) - (s+2) \psi'(s+3) \right) \right. \\
& \quad \left. + 4 \left( (\psi(s+1) + \gamma_E) \psi'(s+1) - \frac{1}{2} \psi''(s+1) \right) - 3 \psi'(s+1) + 4 \psi''(s+1) \right) \\
& + C_A C_F \left( -\frac{1}{(s+1)^3} + \frac{5}{6(s+1)^2} + \frac{53}{18(s+1)} + \frac{\pi^2}{6(s+1)} - \frac{1}{(s+2)^3} \right. \\
& \quad \left. + \frac{5}{6(s+2)^2} - \frac{187}{18(s+2)} + \frac{\pi^2}{6(s+2)} - \frac{67}{9} (\psi(s+1) + \gamma_E) + \frac{1}{3} \pi^2 \right. \\
& \quad \left. (\psi(s+1) + \gamma_E) + 2 \left( \frac{67}{18} - \frac{\pi^2}{6} \right) (\psi(s+1) + \gamma_E) - \frac{11}{3} \psi'(s+1) - \psi''(s+1) \right)
\end{aligned}$$

$$\begin{aligned}
\Phi_{nsq\bar{q}}^{NLO} = & C_F \left( -\frac{C_A}{2} + C_F \right) \left( \frac{2}{(s+1)^3} - \frac{2}{(s+1)^2} + \frac{4}{s+1} - \frac{\pi^2}{3(s+1)} - \frac{1.9968}{(s+2)^3} - \frac{2}{(s+2)^2} \right. \\
& + \frac{3.3246}{s+2} + \frac{3.9404}{(s+3)^3} - \frac{7.1312}{s+3} - \frac{3.602}{(s+4)^3} + \frac{5.8861}{s+4} + \frac{2.6484}{(s+5)^3} + \frac{3.9432}{s+5} - \frac{1.2696}{(s+6)^3} \\
& - \frac{14.24}{s+6} + \frac{0.2796}{(s+7)^3} + \frac{20.43}{s+7} - \frac{19.77}{s+8} + \frac{13.05}{s+9} + \frac{6.286}{s+10} + \frac{1.997}{s+11} - \frac{0.3076}{s+12} \\
& - 2 \left( \frac{4}{(s+1)^3} - \frac{\ln(4)}{(s+1)^2} - \frac{\psi(\frac{s}{2}+1)}{(s+1)^2} + \frac{\psi(\frac{s+1}{2})}{(s+1)^2} + \frac{\psi'(\frac{s}{2}+1)}{2s+2} - \frac{\psi'(\frac{s+1}{2})}{2(s+1)} \right) \\
& - \frac{0.9984}{(s+2)^3} \left( \frac{16}{(s+1)^2} + \frac{12s}{(s+1)^2} + (s+2)\ln(16) - 2(s+2)\psi(\frac{s}{2}+1) + 2(s+1)\psi(\frac{s+1}{2}) \right. \\
& + (s+2)^2\psi'(\frac{s}{2}+1) - (s+2)^2\psi'(\frac{s+1}{2}) \left. \right) - \frac{1.9702}{(s+3)^3} \left( \frac{164}{(s+1)^2(s+2)^2} + \right. \\
& \frac{284s}{(s+1)^2(s+2)^2} + \frac{188s^2}{(s+1)^2(s+2)^2} + \frac{60s^3}{(s+1)^2(s+2)^2} + \frac{8s^4}{(s+1)^2(s+2)^2} - \\
& 4(s+3)\ln(2) - 2(s+3)\psi(\frac{s}{2}+1) + 2(s+3)\psi(\frac{s+1}{2}) + (s+3)^2\psi'(\frac{s}{2}+1) - \\
& (s+3)^2\psi'(\frac{s+1}{2}) \left. \right) - \frac{1.801}{(s+4)^3} \left( \frac{2176}{(s+1)^2(s+2)^2(s+3)^2} + \frac{4392s}{(s+1)^2(s+2)^2(s+3)^2} \right. \\
& + \frac{3504s^2}{(s+1)^2(s+2)^2(s+3)^2} + \frac{1408s^3}{(s+1)^2(s+2)^2(s+3)^2} + \frac{288s^4}{(s+1)^2(s+2)^2(s+3)^2} \\
& + \frac{24s^5}{(s+1)^2(s+2)^2(s+3)^2} + 4(s+4)\ln(2) - 2(s+4)\psi(\frac{s}{2}+1) + 2(s+4)\psi(\frac{s+1}{2}) + \\
& (s+4)^2\psi'(\frac{s}{2}+1) - (s+4)^2\psi'(\frac{s+1}{2}) \left. \right) - \frac{1.3242}{(s+5)^3} \left( \frac{57328}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \right. \\
& \frac{146144s}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{162160s^2}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \\
& \frac{103728s^3}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{42144s^4}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \\
& \frac{11160s^5}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{1880s^6}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \\
& \frac{184s^7}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{8s^8}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} - 4(s+5)\ln(2) \\
& - 2(5+s)\psi(\frac{s}{2}+1) + 2(5+s)\psi(\frac{s+1}{2}) + (s+5)^2\psi'(\frac{s}{2}+1) - (s+5)^2\psi'(\frac{s+1}{2}) \left. \right) - \\
& \frac{0.6348}{(s+6)^2} \left( \ln(16) - 2\psi(\frac{s}{2}+4) + 2\psi(\frac{s+7}{2}) + (s+6)\psi'(\frac{s}{2}+4) - (s+6)\psi'(\frac{s+7}{2}) \right) + \\
& \frac{0.1398}{(s+7)^2} \left( \ln(16) + 2\psi(\frac{s}{2}+4) - 2\psi(\frac{s+9}{2}) - (s+7)\psi'(\frac{s}{2}+4) + (s+7)\psi'(\frac{s+9}{2}) \right) \left. \right)
\end{aligned}$$

$$\begin{aligned}
\Phi_q^{NLO} = & C_F T_f \left( -\frac{40}{9s} + \frac{4}{(s+1)^3} + \frac{28}{3(s+1)^2} - \frac{146}{9(s+1)} + \frac{4}{(s+2)^3} + \frac{52}{3(s+2)^2} + \frac{94}{9(s+2)} + \right. \\
& \frac{16}{3(s+3)^2} + \frac{112}{9(s+3)} + \frac{4}{3}\psi'(s+1) \Big) + C_F^2 \left( \frac{7}{(s+1)^3} + \frac{3}{(s+1)^2} - \frac{1}{s+1} - \frac{\pi^2}{3(s+1)} + \right. \\
& \frac{3.0032}{(s+2)^3} + \frac{1}{(s+2)^2} + \frac{8.3246}{s+2} + \frac{3.9404}{(s+3)^3} - \frac{7.1312}{s+3} - \frac{3.602}{(s+4)^3} + \frac{5.886}{s+4} + \frac{2.6484}{(s+5)^3} \\
& + \frac{3.9432}{s+5} - \frac{1.2696}{(s+6)^3} - \frac{14.2478}{s+6} + \frac{0.2796}{(s+7)^3} + \frac{20.4376}{s+7} - \frac{19.7727}{s+8} + \frac{13.056}{s+9} - \frac{6.2862}{s+10} \\
& + \frac{1.9971}{s+11} - \frac{0.3075}{s+12} - \frac{8}{(s+1)^3} + \frac{2\ln(4)}{(s+1)^2} + \frac{2\psi(\frac{s}{2}+1)}{(s+1)^2} - \frac{2\psi(\frac{s+1}{2})}{(s+1)^2} - \frac{\psi'(\frac{s}{2}+1)}{s+1} + \\
& \frac{\psi'(\frac{s+1}{2})}{(s+1)^2} - \frac{0.9984}{(s+2)^3} \left( \frac{16}{(s+1)^2} + \frac{12s}{(s+1)^2} + (s+2)\ln(16) - 2(s+2)\psi(\frac{s}{2}+1) + \right. \\
& 2(s+2)\psi(\frac{s+1}{2}) + (s+2)^2\psi'(\frac{s}{2}+1) - (s+2)^2\psi'(\frac{s+1}{2}) \Big) - \frac{1.9702}{(s+3)^3} \left( \frac{164}{(s+1)^2(s+2)^2} \right. \\
& + \frac{284s}{(s+1)^2(s+2)^2} + \frac{188s^2}{(s+1)^2(s+2)^2} + \frac{60s^3}{(s+1)^2(s+2)^2} + \frac{8s^4}{(s+1)^2(s+2)^2} - \\
& 4(s+3)\ln(2) - 2(s+3)\psi(\frac{s}{2}+1) + 2(s+3)\psi(\frac{s+1}{2}) + (s+3)^2\psi'(\frac{s}{2}+1) - (s+3)^2\psi'(\frac{s+1}{2}) \Big) \\
& - \frac{1.801}{(s+4)^3} \left( \frac{2176}{(s+1)^2(s+2)^2(s+3)^2} + \frac{4392s}{(s+1)^2(s+2)^2(s+3)^2} + \frac{3504s^2}{(s+1)^2(s+2)^2(s+3)^2} + \right. \\
& \frac{1408s^3}{(s+1)^2(s+2)^2(s+3)^2} + \frac{288s^4}{(s+1)^2(s+2)^2(s+3)^2} + \frac{24s^5}{(s+1)^2(s+2)^2(s+3)^2} + \\
& 4(s+4)\ln(2) - 2(s+4)\psi(\frac{s}{2}+1) + 2(s+4)\psi(\frac{s+1}{2}) + (s+4)^2\psi'(\frac{s}{2}+1) - (s+4)^2\psi'(\frac{s+1}{2}) \Big) \\
& - \frac{1.3242}{(s+5)^3} \left( \frac{57328}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{146144s}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \right. \\
& \frac{162160s^2}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{103728s^3}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \\
& \frac{42144s^4}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{11160s^5}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \\
& \frac{1880s^6}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{184s^7}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \\
& \frac{8s^8}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} - 4(s+5)\ln(2) - 2(s+5)\psi(\frac{s}{2}+1) + 2(s+5)\psi(\frac{s+1}{2}) \\
& + (s+5)^2\psi'(\frac{s}{2}+1) - (s+5)^2\psi'(\frac{s+1}{2}) \Big) - \frac{2}{(s+1)^2} \left( \gamma_E + \frac{1}{s+1} + \psi(s+1) - (s+1)\psi'(s+2) \right) \\
& - \frac{2}{(s+2)^2} \left( \gamma_E + \frac{1}{s+2} + \psi(s+2) - (s+2)\psi'(s+3) \right) - \frac{0.6348}{(s+6)^2} \left( \ln(16) - 2\psi(\frac{s}{2}+4) + \right. \\
& 2\psi(\frac{s+7}{2}) + (s+6)\psi'(\frac{s}{2}+4) - (s+6)\psi'(\frac{s+7}{2}) \Big) + \frac{0.1398}{(s+7)^2} \left( \ln(16) + 2\psi(\frac{s}{2}+4) - \right. \\
& 2\psi(\frac{s+9}{2}) - (s+7)\psi'(\frac{s}{2}+4) + (s+7)\psi'(\frac{s+9}{2}) \Big) +
\end{aligned}$$

Fragmentation Functions of neutral mesons  $\pi^0$  and  $k^0$  with Laplace transform approach 13

$$\begin{aligned}
& C_A C_F \left( \frac{2}{(s+1)^3} + \frac{11}{6(s+1)^2} + \frac{17}{18(s+1)} + \frac{\pi^2}{3(s+1)} - \frac{0.0016}{(s+2)^3} + \frac{11}{6(s+2)^2} - \frac{10.4062}{s+2} \right. \\
& - \frac{1.9702}{(s+3)^3} + \frac{3.5656}{s+3} + \frac{1.801}{(s+4)^3} - \frac{2.9430}{s+4} - \frac{1.3242}{(s+5)^3} - \frac{1.9716}{s+5} + \frac{0.6348}{(s+6)^3} + \frac{7.1239}{s+6} \\
& - \frac{0.1398}{(s+7)^3} - \frac{10.2188}{s+7} + \frac{9.8863}{s+8} - \frac{6.5284}{s+9} + \frac{3.1431}{s+10} - \frac{0.9985}{s+11} + \frac{0.1537}{s+12} - \\
& \frac{67(\psi(s+1) + \gamma_E)}{9} + \frac{1}{3}\pi^2(\psi(s+1) + \gamma_E) + 2 \left( \frac{67}{18} - \frac{\pi^2}{6} \right) (\psi(s+1) + \gamma_E) - \frac{\ln(4)}{(s+1)^2} - \\
& \frac{\psi(\frac{s}{2} + 1)}{(s+1)^2} + \frac{\psi(\frac{s+1}{2})}{(s+1)^2} + \frac{\psi'(\frac{s}{2} + 1)}{2s+2} - \frac{\psi'(\frac{s+1}{2})}{2s+2} + \frac{0.4992}{(s+2)^3} \left( \frac{16}{(s+1)^2} + \frac{12s}{(s+1)^2} + (s+2)\ln(16) \right. \\
& - 2(s+2)\psi(\frac{s}{2} + 1) + 2(s+2)\psi(\frac{s+1}{2}) + (s+2)^2\psi'(\frac{s}{2} + 1) - (s+2)^2\psi'(\frac{s+1}{2}) \Big) + \\
& \frac{0.9851}{(s+3)^3} \left( \frac{164}{(s+1)^2(s+2)^2} + \frac{284s}{(s+1)^2(s+2)^2} + \frac{188s^2}{(s+1)^2(s+2)^2} + \frac{60s^3}{(s+1)^2(s+2)^2} + \right. \\
& \frac{8s^4}{(s+1)^2(s+2)^2} - 4(s+3)\ln(2) - 2(s+3)\psi(\frac{s}{2} + 1) + 2(s+3)\psi(\frac{s+1}{2}) + \\
& (s+3)^2\psi'(\frac{s}{2} + 1) - (s+3)^2\psi'(\frac{s+1}{2}) \Big) + \frac{0.9005}{(s+4)^3} \\
& \left( \frac{2176}{(s+1)^2(s+2)^2(s+3)^2} + \frac{4392s}{(s+1)^2(s+2)^2(s+3)^2} + \frac{3504s^2}{(s+1)^2(s+2)^2(s+3)^2} + \right. \\
& \frac{1408s^3}{(s+1)^2(s+2)^2(s+3)^2} + \frac{288s^4}{(s+1)^2(s+2)^2(s+3)^2} + \frac{24s^5}{(s+1)^2(s+2)^2(s+3)^2} + \\
& 4(s+4)\ln(2) - 2(s+4)\psi(\frac{s}{2} + 1) + 2(s+4)\psi(\frac{s+1}{2}) + (s+4)^2\psi'(\frac{s}{2} + 1) - (s+4)^2\psi'(\frac{s+1}{2}) \Big) \\
& + \frac{0.6621}{(s+5)^3} \left( \frac{57328}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{146144s}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \right. \\
& \frac{162160s^2}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{103728s^3}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \\
& \frac{42144s^4}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{11160s^5}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \\
& \frac{1880s^6}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{184s^7}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \\
& \frac{8s^8}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} - 4(s+5)\ln(2) - 2(s+5)\psi(\frac{s}{2} + 1) + 2(s+5)\psi(\frac{s+1}{2}) + \\
& (s+5)^2\psi'(\frac{s}{2} + 1) - (s+5)^2\psi'(\frac{s+1}{2}) \Big) + \frac{0.3174}{(s+6)^2} \left( \ln(16) - 2\psi(\frac{s}{2} + 4) + 2\psi(\frac{s+7}{2}) + \right. \\
& (s+6)\psi'(\frac{s}{2} + 4) - (s+6)\psi'(\frac{s+7}{2}) \Big) - \frac{0.0699}{(s+7)^2} \left( \ln(16) + 2\psi(\frac{s}{2} + 4) - 2\psi(\frac{s+9}{2}) - \right. \\
& (s+7)\psi'(\frac{s}{2} + 4) + (s+7)\psi'(\frac{s+9}{2}) \Big) - \frac{11}{3}\psi'(s+1) - \psi''(s+1) \Big)
\end{aligned}$$

$$\begin{aligned}
\Theta_q^{NLO} = & T_f^2 \left( \frac{8}{3(s+1)^2} - \frac{40}{9(s+1)} - \frac{16}{3(s+2)^2} + \frac{32}{9(s+2)} + \frac{16}{3(s+3)^2} - \frac{32}{9(s+3)} + \right. \\
& \left. \frac{8(\psi(s+2) + \gamma_E)}{3(s+1)} - \frac{16(\psi(s+3) + \gamma_E)}{3(s+2)} + \frac{16(\psi(s+4) + \gamma_E)}{3(s+3)} \right) + \\
& C_A T_f \left( -\frac{40}{9s} + \frac{4}{(s+1)^3} + \frac{8}{3(s+1)^2} + \frac{26}{9(s+1)} + \frac{24}{(s+2)^3} + \frac{68}{3(s+2)^2} - \frac{33.231}{s+2} \right. \\
& - \frac{4\pi^2}{3(s+2)} + \frac{8}{3(s+3)^2} + \frac{96.875}{s+3} - \frac{67.644}{s+4} + \frac{83.04}{s+5} - \frac{82.976}{s+6} + \frac{56.16}{s+7} - \frac{22}{s+8} \\
& - \frac{22(\psi(s+2) + \gamma_E)}{3(s+1)} + \frac{20(\psi(s+3) + \gamma_E)}{3(s+2)} - \frac{20(\psi(s+4) + \gamma_E)}{3(s+3)} \\
& - 2 \left( \frac{4}{(s+1)^3} - \frac{\ln(4)}{(s+1)^2} - \frac{\psi(\frac{s}{2} + 1)}{(s+1)^2} + \frac{\psi(\frac{s+1}{2})}{(s+1)^2} + \frac{\psi'(\frac{s}{2} + 1)}{2s+2} - \frac{\psi'(\frac{s+1}{2})}{2(s+1)} \right) + \\
& \frac{2}{(s+2)^3} \left( \frac{16}{(s+1)^2} + \frac{12s}{(s+1)^2} + (s+2)\ln(16) - 2(s+2)\psi(\frac{s}{2} + 1) + 2(s+2)\psi(\frac{s+1}{2}) \right. \\
& \left. + (s+2)^2\psi'(\frac{s}{2} + 1) - (s+2)^2\psi'(\frac{s+1}{2}) \right) - \frac{2}{(s+3)^3} \left( \frac{164}{(s+1)^2(s+2)^2} + \frac{284s}{(s+1)^2(s+2)^2} \right. \\
& \left. + \frac{188s^2}{(s+1)^2(s+2)^2} + \frac{60s^3}{(s+1)^2(s+2)^2} + \frac{8s^4}{(s+1)^2(s+2)^2} - 4(s+3)\ln(2) - 2(s+3)\psi(\frac{s}{2} + 1) + \right. \\
& \left. 2(s+3)\psi(\frac{s+1}{2}) + (s+3)^2\psi'(\frac{s}{2} + 1) - (s+3)^2\psi'(\frac{s+1}{2}) \right) + \frac{2(\pi^2 + 6(\psi(s+2) + \gamma_E)^2 - 6\psi'(s+2))}{6s+6} \\
& - \frac{8}{(s+1)^2} \left( \gamma_E + \frac{1}{s+1} + \psi(s+1) - (s+1)\psi'(s+2) \right) - \frac{4(\pi^2 + 6(\psi(s+3) + \gamma_E)^2 - 6\psi'(s+3))}{6s+12} \\
& + \frac{16}{(s+2)^2} \left( \gamma_E + \frac{1}{s+2} + \psi(s+2) - (s+2)\psi'(s+3) \right) + \frac{4(\pi^2 + 6(\psi(s+4) + \gamma_E)^2 - 6\psi'(s+4))}{6s+18} \\
& - \frac{16}{(s+3)^2} \left( \gamma_E + \frac{1}{s+3} + \psi(s+3) - (s+3)\psi'(s+4) \right) \Big) + \\
& C_F T_f \left( -\frac{2}{(s+1)^3} + \frac{7}{(s+1)^2} - \frac{12}{s+1} - \frac{2\pi^2}{3(s+1)} + \frac{4}{(s+2)^3} - \frac{8}{(s+2)^2} + \frac{39.16}{s+2} + \frac{4\pi^2}{3(s+2)} - \frac{8}{(s+3)^3} \right. \\
& - \frac{65.856}{s+3} - \frac{4\pi^2}{3(s+3)} + \frac{77.872}{s+4} - \frac{81.216}{s+5} + \frac{80.128}{s+6} - \frac{51.968}{s+7} + \frac{17.6}{s+8} + \frac{2(\psi(s+1) + \gamma_E)}{s+1} + \\
& \frac{4(\psi(s+2) + \gamma_E)}{s+1} - \frac{4(\psi(s+2) + \gamma_E)}{s+2} + \frac{4(\psi(s+3) + \gamma_E)}{s+3} - \frac{2(\pi^2 + 6(\psi(s+2) + \gamma_E)^2 - 6\psi'(s+2))}{6s+6} \\
& + \frac{12}{(s+1)^2} \left( \gamma_E + \frac{1}{s+1} + \psi(s+1) - (s+1)\psi'(s+2) \right) + \frac{4(\pi^2 + 6(\psi(s+3) + \gamma_E)^2 - 6\psi'(s+3))}{6s+12} \\
& - \frac{24}{(s+2)^2} \left( \gamma_E + \frac{1}{s+2} + \psi(s+2) - (s+2)\psi'(s+3) \right) - \frac{4(\pi^2 + 6(\psi(s+4) + \gamma_E)^2 - 6\psi'(s+4))}{6s+18} + \\
& \left. \frac{24}{(s+3)^2} \left( \gamma_E + \frac{1}{s+3} + \psi(s+3) - (s+3)\psi'(s+4) \right) \right)
\end{aligned}$$

$$\begin{aligned}
\Theta_g^{NLO} = & C_F^2 \left( \frac{2}{(s+1)^3} + \frac{8}{(s+1)^2} - \frac{16.66}{s+1} - \frac{1}{(s+2)^3} - \frac{1}{2(s+2)^2} + \frac{34.196}{s+2} - \frac{40.096}{s+3} + \right. \\
& \frac{42.432}{s+4} - \frac{35.224}{s+5} + \frac{17.392}{s+6} - \frac{4.4}{s+7} - \frac{2(\psi(s+3) + \gamma_E)}{s+2} + \\
& \frac{1}{3s} (\pi^2 + 6(\psi(s+1) + \gamma_E)^2 - 6\psi'(s+1)) - \frac{8}{s^3} (1 + s\gamma_E + s(\psi(s) - s\psi'(s+1))) - \\
& \frac{2}{6s+6} (\pi^2 + 6(\psi(s+2) + \gamma_E)^2 - 6\psi'(s+2)) + \frac{8}{(s+1)^2} \left( \gamma_E + \frac{1}{s+1} + \psi(s+1) - \right. \\
& (s+1)\psi'(s+2)) + \frac{1}{6s+12} (\pi^2 + 6(\psi(s+3) + \gamma_E)^2 - 6\psi'(s+3)) - \\
& \left. \frac{4}{(s+2)^2} \left( \gamma_E + \frac{1}{s+2} + \psi(s+2) - (s+2)\psi'(s+3) \right) \right) + \\
& C_A C_F \left( -\frac{4}{s^3} + \frac{6}{s^2} + \frac{17}{9s} - \frac{2\pi^2}{3s} - \frac{8}{(s+1)^2} + \frac{25.2}{s+1} - \frac{4}{(s+2)^3} - \frac{9}{(s+2)^2} - \frac{23.27}{s+2} - \right. \\
& \frac{\pi^2}{3(s+2)} - \frac{8}{3(s+3)^2} + \frac{35.99}{s+3} - \frac{41.046}{s+4} + \frac{35.01}{s+5} - \frac{17.444}{s+6} + \frac{3.3}{s+7} + \frac{2(\psi(s+3) + \gamma_E)}{s+2} \\
& + \frac{1}{s^2} \left( \ln(16) - 2\psi\left(\frac{s}{2} + 1\right) + 2\psi\left(\frac{s+1}{2}\right) + s\psi'\left(\frac{s}{2} + 1\right) - s\psi'\left(\frac{s+1}{2}\right) \right) - \\
& 2 \left( \frac{4}{(s+1)^3} - \frac{\ln(4)}{(s+1)^2} - \frac{\psi(\frac{s}{2} + 1)}{(s+1)^2} + \frac{\psi(\frac{s+1}{2})}{(s+1)^2} + \frac{\psi'(\frac{s}{2} + 1)}{2s+2} - \frac{\psi'(\frac{s+1}{2})}{2s+2} \right) + \\
& \frac{1}{2(s+2)^3} \left( \frac{16}{(s+1)^2} + \frac{12s}{(s+1)^2} + (s+2)\ln(16) - 2(s+2)\psi\left(\frac{s}{2} + 1\right) + 2(s+2)\psi\left(\frac{s+1}{2}\right) + \right. \\
& (s+2)^2\psi'\left(\frac{s}{2} + 1\right) - (s+2)^2\psi'\left(\frac{s+1}{2}\right) \left. \right) - \frac{1}{3s} \\
& \left( \pi^2 + 6(\psi(s+1) + \gamma_E)^2 - 6\psi'(s+1) \right) + \frac{12}{s^3} (1 + s\gamma_E + s(\psi(s) - s\psi'(s+1))) + \\
& \frac{2}{6s+6} (\pi^2 + 6(\psi(s+2) + \gamma_E)^2 - 6\psi'(s+2)) - \\
& \frac{12}{(s+1)^2} \left( \gamma_E + \frac{1}{s+1} + \psi(s+1) - (s+1)\psi'(s+2) \right) - \\
& \frac{1}{6s+12} (\pi^2 + 6(\psi(s+3) + \gamma_E)^2 - 6\psi'(s+3)) + \\
& \left. \frac{6}{(s+2)^2} \left( \gamma_E + \frac{1}{s+2} + \psi(s+2) - (s+2)\psi'(s+3) \right) \right)
\end{aligned}$$

$$\begin{aligned}
\Phi_g^{NLO} = & C_F T_f \left( -\frac{16}{3s^2} + \frac{92}{9s} + \frac{4}{(s+1)^3} - \frac{10}{(s+1)^2} - \frac{4}{s+1} + \frac{4}{(s+2)^3} - \frac{14}{(s+2)^2} + \frac{12}{s+2} - \frac{16}{3(s+3)^2} - \frac{164}{9(s+3)} \right) \\
& + C_A T_f \left( \frac{8}{3s^2} - \frac{46}{9s} - \frac{4}{(s+1)^2} + \frac{58}{9(s+1)} + \frac{4}{(s+2)^2} - \frac{38}{9(s+2)} - \frac{8}{3(s+3)^2} + \frac{46}{9(s+3)} \right. \\
& \left. + \frac{8}{3}\psi'(s+1) \right) + C_A^2 \left( -\frac{8}{s^3} + \frac{22}{3s^2} + \frac{2}{(s+1)^3} + \frac{11}{(s+1)^2} + \frac{4.4407}{s+1} - \frac{17.9984}{(s+2)^3} + \frac{1}{(s+2)^2} - \right. \\
& \frac{6.9024}{s+2} - \frac{\pi^2}{3(s+2)} + \frac{5.9702}{(s+3)^3} + \frac{22}{3(s+3)^2} - \frac{6.7917}{s+3} + \frac{\pi^2}{3(s+3)} - \frac{1.801}{(s+4)^3} - \frac{3.5389}{s+4} + \\
& \frac{1.3242}{(s+5)^3} + \frac{1.2736}{s+5} - \frac{0.6348}{(s+6)^3} - \frac{5.6479}{s+6} + \frac{0.1398}{(s+7)^3} + \frac{9.2228}{s+7} - \frac{7.6863}{s+8} + \frac{6.5284}{s+9} - \frac{3.1431}{s+10} \\
& \left. + \frac{0.9985}{s+11} - \frac{0.1537}{s+12} - \frac{67(\psi(s+1) + \gamma_E)}{9} + \frac{1}{3}\pi^2(\psi(s+1) + \gamma_E) - \frac{1}{s^2} \left( \ln(16) - 2\psi\left(\frac{s}{2} + 1\right) + \right. \right. \\
& 2\psi\left(\frac{s+1}{2}\right) + s\psi'\left(\frac{s}{2} + 1\right) - s\psi'\left(\frac{s+1}{2}\right) \left. \right) + 2 \left( \frac{4}{(s+1)^3} - \frac{\ln(4)}{(s+1)^2} - \frac{\psi(\frac{s}{2} + 1)}{(s+1)^2} + \frac{\psi(\frac{s+1}{2})}{(s+1)^2} + \right. \\
& \frac{\psi'(\frac{s}{2} + 1)}{2s+2} - \frac{\psi'(\frac{s+1}{2})}{2s+2} \left. \right) - \frac{1.9992}{(s+2)^3} \left( \frac{16}{(s+1)^2} + \frac{12s}{(s+1)^2} + (s+2)\ln(16) - 2(s+2)\psi\left(\frac{s}{2} + 1\right) \right. \\
& \left. + 2(s+2)\psi\left(\frac{s+1}{2}\right) + (s+2)^2\psi'\left(\frac{s}{2} + 1\right) - (s+2)^2\psi'\left(\frac{s+1}{2}\right) \right) + \frac{0.0149}{(s+1)^3} \left( \frac{164}{(s+1)^2(s+2)^2} \right. \\
& \left. + \frac{284s}{(s+1)^2(s+2)^2} + \frac{188s^2}{(s+1)^2(s+2)^2} + \frac{60s^3}{(s+1)^2(s+2)^2} + \frac{8s^4}{(s+1)^2(s+2)^2} - 4(s+3)\ln(2) - \right. \\
& 2(s+3)\psi\left(\frac{s}{2} + 1\right) + 2(s+3)\psi\left(\frac{s+1}{2}\right) + (s+3)^2\psi'\left(\frac{s}{2} + 1\right) - (s+3)^2\psi'\left(\frac{s+1}{2}\right) \left. \right) - \frac{0.9005}{(s+4)^3} \\
& \left( \frac{2176}{(s+1)^2(s+2)^2(s+3)^2} + \frac{4392s}{(s+1)^2(s+2)^2(s+3)^2} + \frac{3504s^2}{(s+1)^2(s+2)^2(s+3)^2} \right. \\
& \left. + \frac{1408s^3}{(s+1)^2(s+2)^2(s+3)^2} + \frac{288s^4}{(s+1)^2(s+2)^2(s+3)^2} + \frac{24s^5}{(s+1)^2(s+2)^2(s+3)^2} + \right. \\
& 4(s+4)\ln(2) - 2(s+4)\psi\left(\frac{s}{2} + 1\right) + 2(s+4)\psi\left(\frac{s+1}{2}\right) + (s+4)^2\psi'\left(\frac{s}{2} + 1\right) - (s+4)^2\psi'\left(\frac{s+1}{2}\right) \left. \right) \\
& - \frac{0.6621}{(s+5)^3} \left( \frac{57328}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{146144s}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} \right. \\
& \left. + \frac{162160s^2}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{103728s^3}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} \right. \\
& \left. + \frac{42144s^4}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{11160s^5}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} \right. \\
& \left. + \frac{1880s^6}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{184s^7}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} + \frac{8s^8}{(s+1)^2(s+2)^2(s+3)^2(s+4)^2} \right. \\
& \left. - 4(s+5)\ln(2) - 2(s+5)\psi\left(\frac{s}{2} + 1\right) + 2(s+5)\psi\left(\frac{s+1}{2}\right) + (s+5)^2\psi'\left(\frac{s}{2} + 1\right) - (s+5)^2\psi'\left(\frac{s+1}{2}\right) \right)
\end{aligned}$$



$$\begin{aligned}
& + \frac{4}{s^3}(1 + \gamma_E s + s(\psi(s) - s\psi'(s+1))) - \frac{8}{(s+1)^2}(\gamma_E + \frac{1}{s+1} + \psi(s+1) - (s+1)\psi'(s+2)) + \\
& \frac{4}{(s+2)^2}(\gamma_E + \frac{1}{s+2} + \psi(s+2) - (s+2)\psi'(s+3)) - \frac{4}{(s+3)^2}(\gamma_E + \frac{1}{s+3} + \psi(s+3) - (s+3)\psi'(s+4)) \\
& - \frac{0.3174}{(s+6)^2} \left( \ln(16) - 2\psi\left(\frac{s}{2} + 4\right) + 2\psi\left(\frac{s+7}{2}\right) + (s+6)\psi'\left(\frac{s}{2} + 4\right) - (s+6)\psi'\left(\frac{s+7}{2}\right) \right) + \\
& \frac{0.0699}{(s+7)^2} \left( \ln(16) + 2\psi\left(\frac{s}{2} + 4\right) - 2\psi\left(\frac{s+9}{2}\right) - (s+7)\psi'\left(\frac{s}{2} + 4\right) + (s+7)\psi'\left(\frac{s+9}{2}\right) \right) \\
& + 4 \left( (\psi(s+1) + \gamma_E)\psi'(s+1) - \frac{1}{2}\psi''(s+1) \right) - \frac{22}{3}\psi'(s+1) + 3\psi''(s+1)
\end{aligned}$$

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